

# Variable Threshold Algorithm for Division of Labor Analyzed as a Dynamical System

Manuel Castillo-Cagigal, Eduardo Matallanas, Iñaki Navarro, Estefanía Caamaño-Martín,  
Félix Monasterio-Huelin and Álvaro Gutiérrez, *Senior Member, IEEE*

**Abstract**—Division of labor is a widely studied aspect of colony behavior of social insects. Division of labor models indicate how individuals distribute themselves in order to perform different tasks simultaneously. However, models that study division of labor from a dynamical system point of view cannot be found in the literature. In this paper, we define a division of labor model as a discrete-time dynamical system, in order to study the equilibrium points and their properties related to convergence and stability. By making use of this analytical model, an adaptive algorithm based on division of labor can be designed to satisfy dynamic criteria. In this way, we have designed and tested an algorithm that varies the response thresholds in order to modify the dynamic behavior of the system. This behavior modification allows the system to adapt to specific environmental and collective situations, making the algorithm a good candidate for distributed control applications. The variable threshold algorithm is based on specialization mechanisms. It is able to achieve an asymptotically stable behavior of the system in different environments and independently of the number of individuals. The algorithm has been successfully tested under several initial conditions and number of individuals.

**Index Terms**—Distributed control, division of labor, dynamical systems, response thresholds, swarm intelligence.

## I. INTRODUCTION

SOCIETIES of insects can perform different tasks simultaneously, distributing the workload among their individual members. This phenomenon is called division of labor and is one of the most basic and widely studied aspects of colony behavior [1], [2]. The parallel task performance can be more efficient than the sequential task performance because the task

switching is avoided, reducing energy and time costs [3], [4]. The division of labor models indicate how individuals are distributed to perform tasks in a way appropriate to the current situation [5]. There are different classes of models based on various hypotheses about the causes of division of labor [6]. This paper focuses on the response threshold models [6]–[8].

Response threshold models operate under the assumption that individuals have internal thresholds associated to stimulus intensities. These stimuli are related to specific tasks and can be any environmental information sensed by individuals e.g., pheromone concentration or the number of encounters with other individuals [1], [9]. The probability of acting or the intensity of an action depends on the stimulus intensities and the response thresholds of the individuals [8]. Every individual has a response threshold for each possible task that it can perform [10]. Variations in stimulus intensities are caused by the task performance of the individuals [11]. Therefore, the action of the individuals modifies and is modified by the stimulus intensities, closing a loop where the stimulus intensities are the system variables.

Although collective systems are commonly studied from the dynamical systems theory perspective [12]–[15], response threshold models of division of labor focus on how the action of individuals performing one or several tasks varies in time. In contrast, this paper focuses on the study of a response threshold model of the division of labor from a dynamical systems theory point of view. This approach allows us to study the system equilibrium points and their properties related to the convergence and stability of the model [16]. In order to carry out this paper, we propose to work on a mono-task situation in discrete-time, where individuals have to decide whether they perform the task or remain inactive. The dynamical system that implements the stimulus intensity loop is based on Bonabeau *et al.*'s proposal [8], [11]. The study of the equilibrium points of the dynamical system is carried out with fixed response thresholds. Hence, the structure of the dynamical system does not vary in time, and the system can be reduced to a single equation.

In previous papers, some authors emphasized that stimuli provided by the environment, as well as individual history, are likely to play an important role in the structure of response threshold models [17]. These studies suggest that the response thresholds should be variable, implementing an adaptive process. According to this suggestion, we propose an algorithm that varies the response thresholds to

Manuscript received January 23, 2013; revised September 19, 2013 and January 15, 2014; accepted February 3, 2014. Date of publication February 28, 2014; date of current version November 13, 2014. This work was supported in part by the Spanish Science and Innovation Ministry under Grant IPT-2011-1468-920000, and in part by the Economy and Competitiveness Ministry under Grant IPT-2012-1072-120000. The work of M. Castillo-Cagigal was sponsored by the Spanish Ministry of Education under Ph.D. Grant FPU-2010 and the work of E. Matallanas was sponsored by the Spanish Ministry of Education under Ph.D. Grant FPU-2011. This paper was recommended by Associate Editor M. Dorigo.

M. Castillo-Cagigal, E. Matallanas, E. Caamaño-Martín, F. Monasterio-Huelin and A. Gutiérrez are with the ETSI Telecomunicación, Universidad Politécnica de Madrid, Madrid 28040, Spain (email: manuel.castillo@upm.es; eduardo.matallanas@upm.es; estefan@ies-def.upm.es; felix.monasteriohuelin@upm.es; aguti@etsit.upm.es).

I. Navarro is with the Distributed Intelligent Systems and Algorithms Laboratory, School of Architecture, Civil and Environment Engineering, École Polytechnique Fédérale de Lausanne, Lausanne 1015, Switzerland and was with the ETSI Telecomunicación, Universidad Politécnica de Madrid during the development of this article (email: inaki.navarro@epfl.ch).

Digital Object Identifier 10.1109/TCYB.2014.2305194

modify the dynamic behavior of the model. This behavior modification allows the system to adapt to specific environmental and collective situations. Several algorithms focused on the modeling of nature to modify the response thresholds can be found in the literature [6], [17]–[20]. Unlike these works, we incorporate concepts of dynamical systems in the design of the variable threshold algorithm. The proposed variable threshold algorithm, based on Theraulaz *et al.*'s proposal [17], is built on a specialization mechanism with a learning and a forgetting coefficient. A specialization mechanism is a reinforcement process where the likelihood to do a task increases when the task is performed and decreases when the task is not performed [1], [21]. We introduce the use of stimulus intensity and individual task performance statistics to retrieve information about the dynamical behavior of the response threshold model. These statistics are included in the proposed algorithm to modify the dynamical behavior of the model, increasing the stability and making the stimulus intensity converge to a desired value. These features have a great importance for the use of response threshold models as distributed controllers [15].

The remainder of this paper is as follows. The response threshold model and the distributed dynamical system are presented in Section II. In this section, we demonstrate the existence of four properties of the dynamical system. In Section III, the variable threshold algorithm is presented. The optimization of the configurable parameters of the variable threshold algorithm and a population study are exposed in Section IV. Finally, the conclusion, potential application, and future work are presented in Section V.

## II. MODEL

In this section, we describe the response threshold model for division of labor for a single task. This model is a discrete-time stochastic process based on the stimulus intensity sensed by the individuals. We propose a transformation of the selected model into an analytical one based on the average action of the individuals. The transformation consists of representing each individual as a deterministic equation included into a discrete-time nonlinear dynamical system. Four properties of the dynamical behavior are deduced from the analytical model. These properties will help in understanding the development of the variable threshold algorithm explained in Section III.

### A. Response Threshold Model

Response threshold models postulate that a task is performed in response to an associated stimulus intensity  $s \in \mathbb{R}$ . Individuals increase their likelihood of performing a task when  $s$  exceeds their internal response threshold  $\Theta \in \mathbb{R}$ . These models are stochastic processes because the task performance depends on the probability of acting. The relationship between the stimulus intensity and the number of individuals performing the task is inversely proportional. Therefore, response threshold models implement a negative feedback loop, where  $s$  is the system variable. This study is done in the discrete-time framework and the time is denoted with subscript  $k$ . The

evolution of  $s_k$  regulates the number of individuals working on a task [22].

In this model, the performance of individual  $i$  with respect to a task at instant  $k$  is defined as a binary process represented by  $f_k^i$ . Where  $f_k^i = 0$  if the individual  $i$  is not working on the task at instant  $k$  and  $f_k^i = 1$  on the contrary. It implies that there are not performance differences between individuals. At each instant, each individual works on the task with probability  $P(f_k^i = 1) = p_k^i$ . Therefore,  $f_k^i$  is a Bernoulli process, where the probability of acting of individual  $i$  is a function of the  $s_k$  associated with the task. Hence

$$p_k^i = T(s_k, \Theta^i) \quad (1)$$

where  $\Theta^i$  is the response threshold of individual  $i$  and  $T(\cdot)$  is the response threshold function common to all individuals.

The response threshold functions cannot take any possible shape; they are probability functions and should represent the relationship between  $s_k$  and  $p_k^i$ , as previously explained. We define the following constraints for the response threshold functions: 1)  $T(\cdot) \in [0, 1]$ , representing the probability of acting; 2)  $T(\cdot)$  is a monotonically increasing function, the higher the probability of acting the higher  $s$ ; 3) it must be continuous and differentiable in  $\mathbb{R}$ ; 4) the higher the slope of  $T(\cdot)$  ( defined as  $T' = dT/ds$  ) the closer  $\Theta^i$  to  $s$ . The last constraint involves the fact that individuals become insensitive to stimulus intensity variations far from their response thresholds. Logistic functions (e.g., Sigmoid) are examples of response threshold functions that satisfy the four constraints mentioned above.

As aforementioned, the stimulus intensity is modified by the individual task performance. The environment also produces variations in the stimulus intensity independently of whether or not the task is performed. Then, the resulting equation for discrete-time dynamics of  $s_k$  is given as follows:

$$s_{k+1} = D \cdot s_k + \delta - N_k^{act} \quad (2)$$

where  $D$  and  $\delta$  are two constants which represent two environmental parameters.  $D$  represents the inertia of the environment and  $\delta$  a continuous increment of the stimulus intensity. The environment is considered to be stable and autonomous; hence  $D \in [0, 1)$  and  $\delta \in \mathbb{R}$ .  $N_k^{act} \in \mathbb{N}$  is the number of individuals performing the task at instant  $k$  and it is defined as

$$N_k^{act} = \sum_{i=1}^N f_k^i \quad (3)$$

where  $N \in \mathbb{N}$  is the number of individuals.  $N_k^{act}$  introduces the nonlinearity in (2).  $N_k^{act}$  is a random process because it comprises the individual actions that are themselves random processes as discussed above. In summary, (2) is the discrete-time stochastic dynamical system which defines the response threshold model of the division of labor.

### B. Analytical Model

With the aim of analyzing the dynamics of the response threshold model, it is necessary to approximate (2) by an analytical equation, eliminating the random term. If  $N_k^{act}$  is approximated by its expected value, the sum of individuals

task performance is approximately the sum of probabilities of acting. By combining (1), (2), and (3), the analytical model is defined as

$$N_k^{act} \approx E[N_k^{act}] = \sum_{i=1}^N T(s_k^*, \Theta^i) = F(s_k^*, \Theta) \quad (4)$$

$$s_{k+1}^* = D \cdot s_k^* + \delta - F(s_k^*, \Theta) \quad (5)$$

where  $s_k^* \in \mathbb{R}$  is the stimulus intensity at instant  $k$  for the analytical model,  $\Theta$  is the vector of all individual response thresholds, and  $F(s^*, \Theta)$  is the sum of all response threshold functions. The analytical model is a discrete-time deterministic dynamical system where the system variable is  $s^*$ .

### C. Response Threshold Model Dynamics

The study of the response threshold dynamics allows understanding the evolution of the stimulus intensity for a certain configuration of the dynamical system. A configuration is defined as a fixed set of values for the environmental parameters  $D$  and  $\delta$ , the response threshold functions and the response thresholds, respectively. Depending on the configuration of the dynamical system,  $s_k^*$  can converge to a single value, perform an oscillatory behavior or respond differently to disturbances. To design the variable threshold algorithm, described in Section III, it is mandatory to define some dynamic properties of the analytical model. These properties are defined hereafter.

In the analytical model,  $s_k^*$  can be represented as a discrete sequence for an initial stimulus intensity  $s_0^*$

$$\{S\} : s_0^*, s_1^* = H(s_0^*), \dots, s_k^* = H(s_{k-1}^*) \quad (6)$$

$$H(s^*) = D \cdot s^* + \delta - F(s^*, \Theta) \quad (7)$$

where  $H(s^*)$  is the dynamic function of (5).  $H(s^*)$  can be plotted as a stair-step diagram, where the y-axis is the stimulus intensity in  $k+1$ , while the x-axis is the stimulus intensity in  $k$ . An example of a generic  $H(s^*)$  function in a stair-step diagram is shown in Fig. 1(a). The evolution of  $s_k^*$  can be easily calculated using this diagram by the projections of  $H(s_k^*)$  in the bisectrix,  $s_{k+1} = s_k$ , and back to  $H(s_{k+1}^*)$ . This graphical method is shown in Fig. 1(a), where the discrete sequence  $\{s_0^*, s_1^*\}$  is represented by dotted arrows.

The notion of equilibrium points is the central problem in the stability of dynamical systems [16]. The evolution of  $s_k^*$  can be defined in relation to its equilibrium points. A point  $\hat{s}$  of the nonlinear dynamic system  $s_{k+1}^* = H(s_k^*)$  is an equilibrium point if it is a fixed point of the system ( $\hat{s} = H(\hat{s})$ ). Geometrically, an equilibrium point is the intersection between the function  $s_{k+1}^* = H(s_k^*)$  and the bisectrix,  $s_{k+1}^* = s_k^*$ . It is a constant solution because  $s_{k+1}^* = H(s_k^*) = s_k^*$ ,  $\forall k$  if  $s_0^* = \hat{s}$ . The equilibrium point  $\hat{s}$  of  $H(s^*)$  can be observed in Fig. 1(a). Once the equilibrium points have been defined, two properties can be established.

*Property 1:* The slope of  $H(s)$ , defined as  $H'(s) = dH(s)/ds$ , is in the range  $(-\infty, 1)$ .

*Proof:* The function  $H(s)$  is the sum of two terms which depend on  $s$  (7), hence  $H'(s) = D - F'(s, \Theta)$ .  $F'(s, \Theta) \in [0, \infty)$  because  $D \in [0, 1)$  and  $F(s, \Theta)$  is a continuous monotonically

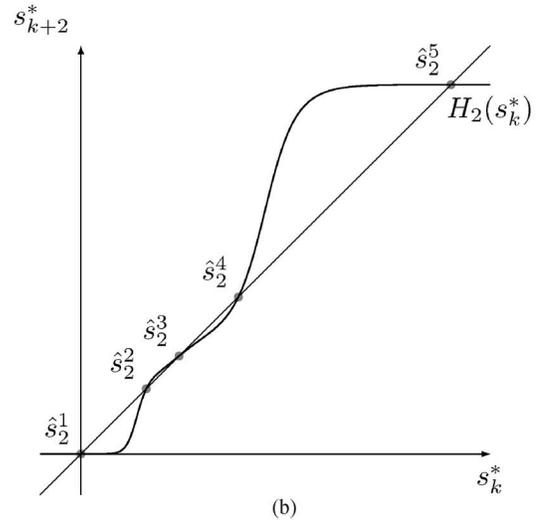
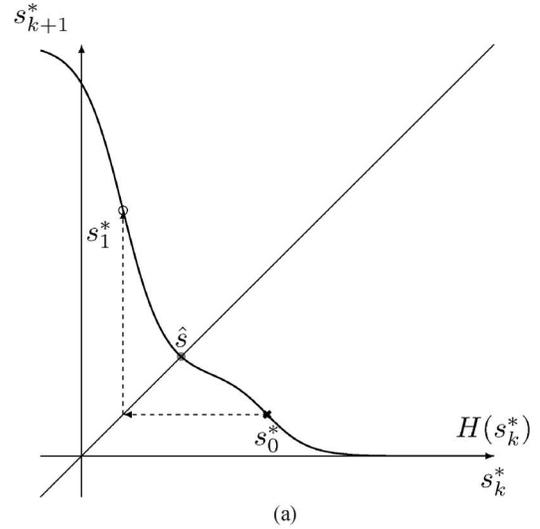


Fig. 1. Stair-step diagrams for a generic  $H(s^*)$  function with  $D = 0$  and  $N = 100$ . (a) First iteration, where  $\hat{s}$  is the unique constant equilibrium point and the dotted arrows represent two steps of a stimulus intensity sequence  $\{s_0^*, s_1^*\}$ . (b) Five periodic equilibrium points of the second iteration.

increasing function. Therefore, it is satisfied that  $H'(s) \in (-\infty, 1)$ . ■

*Property 2:* If it exists, there is only one equilibrium point of  $H(s)$ .

*Proof:* Let  $\hat{s}_1$  be an equilibrium point of  $H(s)$ . It is supposed that another equilibrium point  $\hat{s}_2$  exists such that  $\hat{s}_1 \neq \hat{s}_2$ . Notice that,  $H(s)$  is continuous and differentiable because  $F(s, \Theta)$  is the sum of  $T(s, \Theta^i)$  which are continuous and differentiable too. Therefore, according to the Mean Value Theorem, it exists an  $s_a \in (\hat{s}_1, \hat{s}_2)$  such that  $H'(s_a) = (H(\hat{s}_2) - H(\hat{s}_1)) / (\hat{s}_2 - \hat{s}_1) = 1$  which contradicts Property 1. Therefore, only one equilibrium point of  $H(s)$  is possible. ■

However, the restriction of a unique constant solution does not limit the existence of multiple periodic stimulus intensity behaviors. These periodic behaviors are called periodic orbits of period  $T$ . The orbits are composed by a sequence of  $T$  stimulus intensity values repeating permanently. In this situation, it is satisfied that  $s_{k+T}^* = s_k^*$ . These periodic orbits

cannot be directly calculated from  $H(s^*)$ , but from the analysis of  $H_T(s^*)$ , being the  $T$ -iteration of the difference equation ( $H_T = H \circ \dots \circ H$ ). Notice that  $s_{k+T}^* = H_T(s_k^*)$  represents the sequence  $\{S_T\} : \{s_0^*, s_T^*, s_{2T}^*, \dots\}$  and it does not include the elements of the  $\{S\}$  sequence between  $s_{k+T}^*$  and  $s_k^*$ . The concept of equilibrium point can also be applied to  $H_T(s^*)$ ; hence a periodic orbit is a fixed point of the iterated difference equation ( $\hat{s}_T^j = H_T(\hat{s}_T^j)$ ). It can also be represented graphically, plotting the function  $s_{k+T}^* = H_T(s_k^*)$  and the bisectrix  $s_{k+T}^* = s_k^*$ . The equilibrium points of the second iteration of a generic function  $H_2(s^*)$  can be observed in Fig. 1(b). In this case, the  $y$ -axis is the  $k + 2$  stimulus intensity. Notice that Properties 1 and 2 cannot be applied generally to  $H_T(s^*)$ , because the slope of  $H_T(s^*)$  is not necessarily in the range  $(-\infty, 1]$ .

To know the existence and location of the equilibrium point is not enough to understand the response threshold model dynamics. An equilibrium point has different dynamic properties based on the stability when  $s_k^*$  takes a value close to it. An equilibrium point  $\hat{s}$  is an attractor when  $s_k^* \rightarrow \hat{s}$  for  $k \rightarrow \infty$ ; hence,  $s_k^*$  converges to this equilibrium point. An example of attractor is the equilibrium point  $\hat{s}$  of Fig. 2(a). The evolution of the stimulus intensity in the analytical model can be observed using the stair-step diagram. For a  $s_k^*$  value close to  $\hat{s}$ ,  $s_k^*$  converges to it. The attractor concept can also be applied to periodic orbits. In Fig. 2(c),  $s_k^*$  converges to an oscillatory behavior for a value close to the periodic orbits of period  $T = 2$ . The attractors do not only appear in the analytical model, but also in the stochastic one. As for the randomness of the process,  $s_k$  does not converge to a constant value and its variance depends on the individuals probability of acting. The evolution of  $s_k$  can be observed in Fig. 2(b) and (d). In this case, it converges to the attractors of the analytical model. Therefore, it is important to analyze if the equilibrium points of the system are attractors or not, as defined in Property 3.

*Property 3:* If  $\hat{s}$  exists and it is satisfied that  $|\Delta F(s_k, \Theta)| < (1 + D)|\Delta s_k|$ ;  $\forall k > M \in \mathbb{N}$  then  $\hat{s}$  is an attractor. Where  $\Delta s_k = s_{k+1} - s_k$  and  $\Delta F(s_k, \Theta) = F(s_{k+1}, \Theta) - F(s_k, \Theta)$ .

*Proof:* If  $|\Delta s_{k+1}| < |\Delta s_k|$ ;  $\forall k > M \in \mathbb{N} \Rightarrow \exists \epsilon > 0$  such that  $|\Delta s_{k+1}| < \epsilon$ ;  $\forall k > N > M$ . Hence, sequence  $\{S\}$  is a Cauchy sequence, being it convergent. By using (6) and (7), the previous sequence can be rewritten as

$$|D \cdot \Delta s_k - \Delta F(s_k, \Theta)| < |\Delta s_k|; \forall k > M$$

As  $F(s_k, \Theta)$  is a monotonically increasing function,  $\Delta s_k > 0 \Leftrightarrow \Delta F(s_k, \Theta) > 0$ ; hence, the previous inequality is satisfied if

$$(D - 1)|\Delta s_k| < |\Delta F(s_k, \Theta)| < (1 + D)|\Delta s_k|$$

The lower limit is negative because  $D \in [0, 1)$ , and the previous equation can be rewritten as

$$|\Delta F(s_k, \Theta)| < (1 + D)|\Delta s_k|$$

Therefore,  $\{S\}$  is a Cauchy sequence if  $|\Delta F(s_k, \Theta)| < (1 + D)|\Delta s_k|$ ;  $\forall k > M$ . According to Property 2, there is only one equilibrium point  $\hat{s}$ , so  $\{S\}$  sequence converges to  $\hat{s}$ , which is an attractor. ■

The convergence of sequence  $\{S\}$  to  $\hat{s}$  is linked to the distance between  $s_0^*$  and  $\hat{s}$ . This distance is related to the concept of domain of attraction  $A(\hat{s}) \subset \mathbb{R}$ . If  $s_0 \in A(\hat{s})$  then it is satisfied that  $s_k \rightarrow \hat{s}$  when  $k \rightarrow \infty$ . The sequence  $\{S\}$ , which satisfies Property 3, belongs to  $A(\hat{s})$ . An example of the domain of attraction is shown in Fig. 2(a) and (c), where  $s_0 \in A(\hat{s})$  and  $s_0 \notin A(\hat{s})$ , respectively. The calculation of the domain of attraction is usually either very complex or has even no analytical solution. In other cases, as the ones shown in Fig. 2(a) and (c), an analytical solution is found by means of Property 4.

*Property 4:* If  $D = 0$ , the domain of attraction of  $\hat{s}$  of  $H(s)$  is limited by the next higher and next lower equilibrium points of  $H_2(s)$ .

*Proof:* If  $D = 0$ , according to (7)

$$H(s) = \delta - F(s, \Theta)$$

which is a monotonically decreasing function.  $H'(s) \in (-\infty, 0]$ ,  $\forall s \in \mathbb{R}$ , with an unique equilibrium point  $\hat{s}$  (see Property 2). According to the Chain Rule,  $H'_m(s) = H'(H_{m-1}(s)) \cdot H'_{m-1}(s)$ . For this reason, it is satisfied that all  $H_m(s)$  functions, with  $m$  being an odd number, are monotonically decreasing, with one single equilibrium point. All  $H_m(s)$  functions, with  $m$  being an even number, are monotonically increasing. In this case, it is satisfied that  $\hat{s}_2^i < H_2(s) < H_m(s) < \hat{s}_2^{i+1}$ . Therefore, there are no equilibrium points between  $[\hat{s}_2^i, \hat{s}_2^{i+1}]$  and it implies that the equilibrium points of  $H_m(s)$  are the same as those of  $H_2(s)$ . If  $\hat{s}$  is an attractor, then  $0 \leq H'_2(\hat{s}) < 1$ . Let  $s^+$  and  $s^-$  be the next higher and next lower equilibrium point of  $H_2(s)$ , respectively. They are unstable and for each  $s_0 \in (s^-, s^+)$ ,  $|s_k - \hat{s}| \rightarrow 0$  when  $k \rightarrow \infty$ . Therefore, the domain of attraction of  $\hat{s}$  of  $H(s)$  when  $D = 0$  is limited by  $s^+$  and  $s^-$ . ■

Once  $s_k^*$  has converged to an equilibrium point or periodic orbit, it stays in this state indefinitely. The response threshold model dynamics can vary the equilibrium point in which it is operating if  $s_k^*$  takes a value in other domain of attraction. This  $s_k^*$  modification can be obtained through an external disturbance. This event would modify the stimulus intensity evolution permanently.

To represent the disturbances mathematically, we use the nonlinear function Kronecker delta  $\delta_{k-l}$ . This function is 1 if  $k = l$ , and 0 otherwise.  $\delta_{k-l}$  is a nonpermanent stimulus intensity disturbance which occurs at time instant  $l$ . It is inserted in the response threshold model dynamics as follows:

$$s_{k+1}^* = H(s_k^k) + a_l \cdot \delta_{k-l} \quad (8)$$

where  $a_l$  is the disturbance amplitude at instant  $l$ .  $s_k^*$  can converge to another equilibrium point depending on  $a_l$  and the size of the domain of attraction of the equilibrium point in which the response threshold dynamics is operating. An example with two disturbances can be observed in Fig. 3(a). In this example,  $s_0^*$  is in the domain of attraction of  $\hat{s}$ ; hence,  $s_k^*$  converges to this equilibrium point. The first disturbance takes place at  $k = 50$  and it displaces the stimulus intensity out of  $A(\hat{s})$ . This behavior causes that the stimulus intensity diverges from  $\hat{s}$  and converges to a periodic orbit. The second

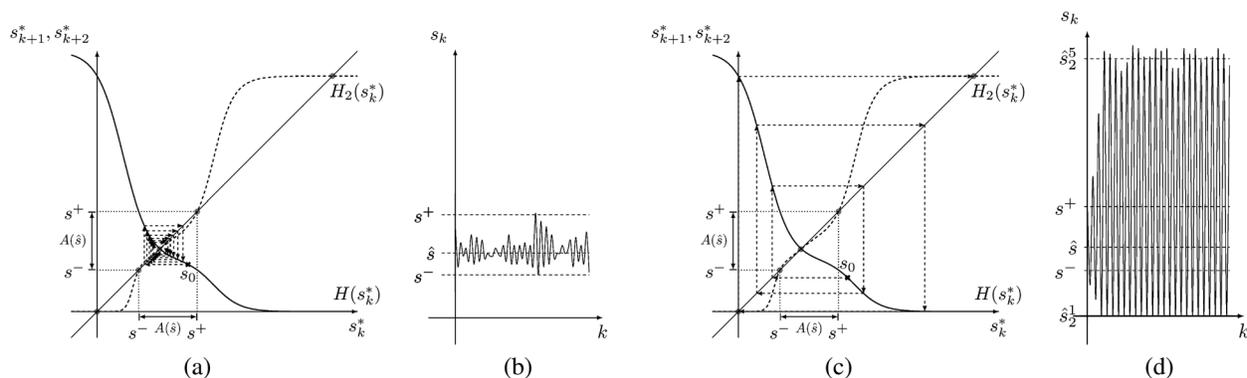


Fig. 2. Evolution of stimulus intensity for a generic  $H(s^*)$  function with  $D = 0$  and  $N = 100$  and for different initial values. (a) Stair-step diagram for  $s_0 \in A(\hat{s})$ . (b) Evolution of the stochastic model for  $s_0 \in A(\hat{s})$ . (c) Stair-step diagram for  $s_0 \notin A(\hat{s})$ . (d) Evolution of the stochastic model for  $s_0 \notin A(\hat{s})$ . Domains of attraction are represented by their upper ( $s^+$ ) and lower ( $s^-$ ) values.

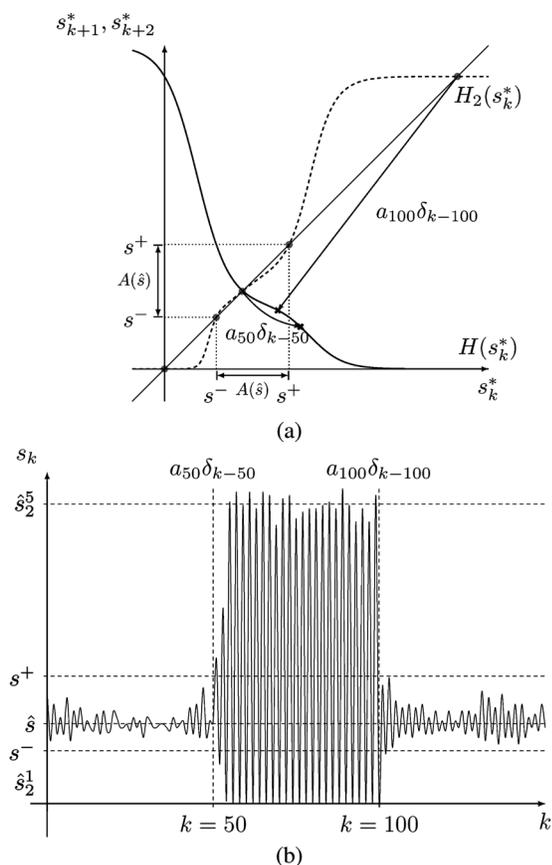


Fig. 3. Disturbance effects example with two disturbances  $a_{50} \cdot \delta_{k-50}$  and  $a_{100} \cdot \delta_{k-100}$  for a generic  $H(s^*)$  function with  $D = 0$  and  $N = 100$ . (a) Representation of disturbances in the analytical model. (b) Evolution of the stimulus intensity in the stochastic model.

disturbance takes place at  $k = 100$  and it returns the stimulus intensity to the domain of attraction of  $\hat{s}$ . The evolution of  $s_k$  for the stochastic model is shown in Fig. 3(b) with similar dynamic properties to the analytical one.

### III. VARIABLE THRESHOLD ALGORITHM

A dynamical system can be designed to perform specific dynamic behaviors, e.g., ensure that  $s_k$  converges asymptotically to a concrete value. The downside of this procedure

is that it is necessary to know the environmental parameters when the system is designed. If environmental parameters are not known, the dynamical behavior of the response threshold model cannot be ensured. In this case, the response thresholds should be modified in real time, adapting to the environment. In this section, we propose a variable threshold algorithm which ensures the convergence of the stimulus intensity to a certain value without the need to know the environmental parameters. The variable threshold algorithm modifies the response thresholds in a distributed manner. There is no explicit information exchange between individuals. Through the evaluation of their own work and  $s_k$ , the individuals modify their response thresholds and hence, the dynamical system behavior.

The variable threshold algorithm takes into account the response threshold dynamics properties and has the following features.

- 1) *Stability against disturbances*: The presence of stimulus intensity disturbances ( $a_k \cdot \delta_k$ ) can modify the equilibrium point in which the dynamical system is operating. It implies that  $s_k$  evolution varies in a qualitative point of view, e.g., from a nonoscillatory behavior to an oscillatory one. The objective of the variable threshold algorithm is to stabilize the response threshold dynamics in a concrete equilibrium point, in such a way that the disturbances cannot get  $s_k$  out of it.
- 2) *Oscillatory behavior avoidance*: If the system is operating in a periodic orbit,  $s_k$  varies periodically. The objective of the variable threshold algorithm is to converge asymptotically to a concrete value and therefore oscillatory behaviors must be avoided.
- 3) *Reference stimulus intensity*: Although  $s_k$  converges asymptotically to a stable equilibrium point, the value of this equilibrium point is not guaranteed. The objective of the variable threshold algorithm is to make  $s_k$  converge to a reference value.

#### A. Stability Against Disturbances

Every dynamical system that faces a real environment must take into account the presence of disturbances. Variations of  $s_k$  induced by disturbances can modify the evolution of  $s_k$  permanently, as explained in Section II. This effect should

be avoided by the variable threshold algorithm. Permanent modifications of  $s_k$  evolution arise when  $s_k$  takes a value in a domain of attraction of another equilibrium point. Hence, the permanent modifications induced by disturbances can be avoided if the domain of attraction, where the stimulus intensity is located, increases.

*Conjecture 1:* The farther the response thresholds from  $\hat{s}$  are, the greater the domain of attraction of  $\hat{s}$  is.

*Reasoning:* As for the definition of the response threshold function,  $T'(\hat{s}, \Theta^i)$  is reduced in a region close to  $\hat{s}$  if the distance between  $\Theta^i$  and  $\hat{s}$  is increased.  $F'(\hat{s}, \Theta)$  is reduced if the response thresholds increase their distance from  $\hat{s}$ , because  $F(s, \Theta)$  is the sum of the individual response threshold functions. It implies that the difference  $\Delta F(s_k, \Theta)$  for two points  $s_k$  and  $s_{k+1}$  that belongs to a region close to  $\hat{s}$ , decreases as far as the response thresholds from  $\hat{s}$  are. Therefore, if the response thresholds tend to increase their distances with respect to  $\hat{s}$ , the region in which the sequence  $\{S_k\}$  satisfies the inequality of Property 3 increases.

In order to implement Conjecture 1, the variable threshold algorithm is based on a specialization mechanism. Specialization is a postulated mechanism in which the more often an individual performs a task, the more often it continues doing the same task with a higher probability [17]. In a specialization process, some individuals tend to do always the same task while others never do it ( $p_k^i \approx 1$  or  $p_k^i \approx 0$ ,  $\forall k$  respectively); when this occurs, individuals are specialized. In the response threshold models, the response threshold decreases when the corresponding task is performed and vice versa, implementing a positive feedback loop and increasing the distance between the response thresholds and  $\hat{s}$ . The response threshold variation  $\Delta\Theta_k^i$  of the individual  $i$  at instant  $k$  is defined by the following nonlinear equations

$$\Theta_{k+1}^i = \Theta_k^i + \Delta\Theta_k^i \quad (9)$$

$$\Delta\Theta_k^i = \begin{cases} -\varphi_k^i + \xi_k^i & \text{if } f_k^i = 1 \\ \varphi_k^i - \xi_k^i & \text{if } f_k^i = 0 \end{cases} \quad (10)$$

where  $\varphi_k^i$  is the learning coefficient which implements the specialization and  $\xi_k^i$  is the forgetting coefficient which implements the opposite process to specialization. Note that the response threshold modification depends on the task performance of individual  $i$ . It implies that if an individual works on the task, its response threshold is reduced by increasing its probability of acting. On the contrary, if an individual does not work on the task, its response threshold is increased reducing its probability of acting.

When an individual is specialized, the learning coefficient continues increasing (or decreasing) the response threshold indefinitely, so it satisfies  $|\Theta_k^i - s_k| \rightarrow \infty$  when  $k \rightarrow \infty$ . According to the definition of the response threshold function, it implies that  $T'(s_k, \Theta_k^i) \rightarrow 0$ . Therefore, the individuals become insensible to  $s_k$  variations and continue doing always the same action; this phenomenon is called overspecialization. This situation has been solved in previous papers limiting the response threshold values in a certain range. In this paper, we propose the use of a continuous function to control the

response threshold variations. The necessary information to perform this feature can be locally obtained from the following two stimulus intensity statistics:  $S_k$  that is the exponential moving average of  $s_k$  and  $\sigma_k$  that is the exponential moving variance of  $s_k$ . Hence

$$S_k = \rho s_k + (1 - \rho)S_{k-1} \quad (11)$$

$$\sigma_k = \rho(s_k - S_k)^2 + (1 - \rho)\sigma_{k-1} \quad (12)$$

Both time-dependent variables take into account the current stimulus value and its previous history.  $\rho$  is a configurable parameter which represents the weight of new samples in relation with the historical one. As  $\rho \in [0, 1]$ , the higher  $\rho$  is, the more relevant current values are and vice versa. Then, a new property can be enunciated to define the control function as:

*Property 5:* When the stimulus intensity has converged to  $\hat{s}$ , the lower the variance is, the greater the distance between response thresholds and  $\hat{s}$  is.

*Proof:* Consider the variance of  $N_k^{act}$  as

$$Var[N_k^{act}] \approx \sum_{i=1}^N p_k^i(1 - p_k^i) = \sum_{i=1}^N T(s_k, \Theta^i)(1 - T(s_k, \Theta^i))$$

where the definition of the response threshold function,  $T(\cdot) \rightarrow 0$  or  $1$  when  $|s_k - \Theta^i| \rightarrow \infty$  and  $T(\cdot) \rightarrow 0.5$  when  $|s_k - \Theta^i| \rightarrow 0$ . Therefore,  $T(\cdot)(1 - T(\cdot)) \rightarrow 0$  when  $|s - \Theta^i| \rightarrow \infty$  and  $T(\cdot)(1 - T(\cdot)) \rightarrow 0.25$  when  $|s - \Theta^i| \rightarrow 0$ , which is the maximum value. Let  $\Omega$  be a vector of response thresholds  $\Theta^i \in \Omega$  and  $\Phi$  another vector of response thresholds  $\Theta^j \in \Phi$  such that  $|\hat{s} - \Theta^i| < |\hat{s} - \Theta^j|$ . Therefore,  $T(s_k, \Theta^i)(1 - T(s_k, \Theta^i)) > T(s_k, \Theta^j)(1 - T(s_k, \Theta^j)) \forall i$  and  $Var[N_k^{act}]_{\Omega} > Var[N_k^{act}]_{\Phi}$ . It means that the greater the distance of the response threshold to the equilibrium point is, the lower the variance is. ■

Conjecture 1 together with Property 5 establish  $\sigma_k$  as an indicator of the domain of attraction and the overspecialization of the individuals. If the response thresholds are close to the equilibrium point,  $\sigma_k$  increases. On the other hand, if the individuals are overspecialized then  $\sigma_k \rightarrow 0$ . The variable threshold algorithm has to reach a compromise between the stability and the overspecialization. Following this compromise,  $\varphi_k$  and  $\xi_k$  can be defined as

$$\varphi_k = \nu_k \cdot e^{-\frac{|s_k - \Theta_k^i|}{\nu_k}} \quad (13)$$

$$\xi_k = e^{\alpha_2 \sigma_k} \quad (14)$$

$$\nu_k = (1 + \alpha_1 \sqrt{\sigma_k}) \quad (15)$$

where  $\nu_k$  denotes the standard deviation of  $s_k$ . Note that  $\varphi_k$  decreases when the distance between the stimulus intensity and the threshold value increases; hence, it avoids overspecialization. The higher  $\nu_k$  is, the higher the distance between  $s_k$  and  $\Theta_k^i$  is, because the exponential index of  $\varphi_k$  is inversely proportional to  $\nu_k$ . Moreover, the magnitude of the learning coefficient is directly proportional to  $\nu_k$  which implies that the higher  $\sigma_k$  is, the higher the learning step is.  $\alpha_1$  is a configurable

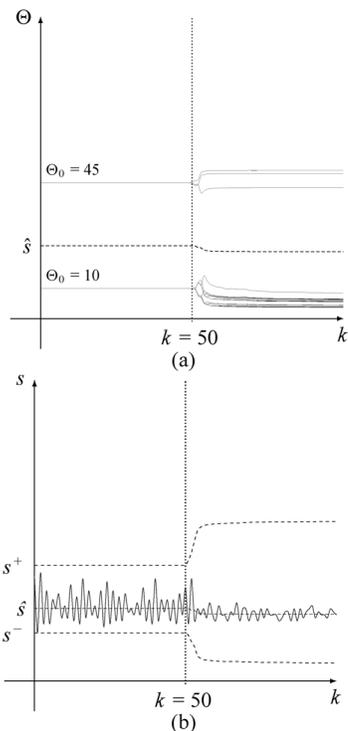


Fig. 4. Variable threshold algorithm example for  $s_0 \in A(\hat{s})$  with  $D = 0$ ,  $\delta = 100$ , and  $N = 100$  with 80% of individuals having  $\Theta_0 = 10$  and 20% having  $\Theta_0 = 45$ . (a) Evolution in time of response thresholds. (b) Evolution of stimulus intensity and domain of attraction limits ( $s^-$ ,  $s^+$ ).

parameter which regulates the effect of  $\sigma_k$  in  $\varphi_k$ . The forgetting coefficient is the opposite to the learning one; hence, the higher this term is the lower  $\sigma_k$  is, avoiding overspecialization.  $\alpha_2$  is a configurable parameter which regulates the effect of  $\sigma_k$  in  $\xi_k^i$ .

An example of the variable threshold algorithm is shown in Fig. 4. In this example, the environmental parameters are  $D = 0$  and  $\delta = 100$ . The population of individuals is  $N = 100$ , where 80 individuals have an initial response threshold  $\Theta_0^i = 10$  and 20 individuals  $\Theta_0^i = 45$ . The  $H(s, \Theta)$  function of the dynamical system was already plotted in Fig. 1(a).  $s_0$  is in the domain of attraction of the asymptotically stable equilibrium point ( $s_0 \in A(\hat{s})$ ). During the first time steps ( $k < 50$ ), the variable threshold algorithm is not acting; hence, the response thresholds maintain the same value as observed in Fig. 4(a). In the time step  $k = 50$ , the variable threshold algorithm is activated and it modifies the response thresholds. The distance between the response thresholds and the asymptotically stable equilibrium point is increased, because of the variance of the stimulus intensity. Moreover,  $\sigma_k$  is reduced and the domain of attraction is increased, as can be observed in Fig. 4(b).

### B. Oscillatory Behavior Avoidance

We propose another example to show the effect of oscillatory behavior in Fig. 5. The environmental parameter, the number of individuals, and the response thresholds initial values are the same as those in the previous experiment. In this example, the initial stimulus intensity is not in the domain of attraction of the asymptotically stable equilibrium point ( $s_0 \notin A(\hat{s})$ ). As in the previous example, the variable threshold

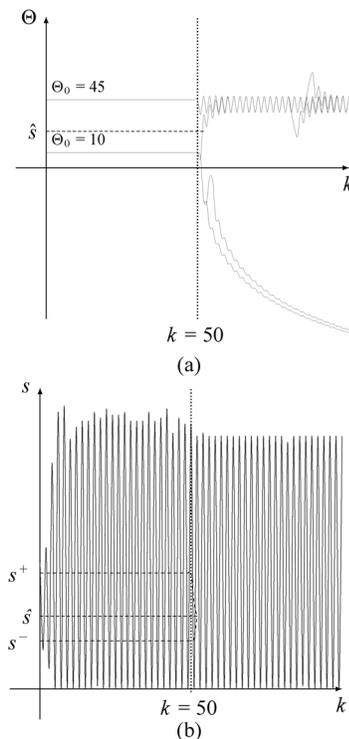


Fig. 5. Variable threshold algorithm example for  $s_0 \notin A(\hat{s})$  with  $D = 0$ ,  $\delta = 100$ , and  $N = 100$  with 80% of individuals having  $\Theta_0 = 10$  and 20% having  $\Theta_0 = 45$ . (a) Evolution in time of response thresholds. (b) Evolution of stimulus intensity and domain of attraction limits ( $s^-$ ,  $s^+$ ).

algorithm is activated at time step  $k = 50$ . In this situation, the algorithm cannot drive  $s_k$  to a nonoscillatory solution [see Fig. 5(b)]. Fig. 5(a) shows how the response threshold values are modified, performing an oscillatory behavior.

An oscillatory behavior implies that several individuals are changing permanently their task performance  $f_k^i$ . Individuals that carry out these oscillations are sensing different stimulus intensity values, lower and higher than their response thresholds, and they provoke that the probability of acting switches between a high and a low value. Therefore, the response thresholds of the individuals which are performing an oscillatory behavior are between the stimulus intensity values.

As shown in the previous example, the specialization mechanism is not able to avoid oscillations in its actual form. Oscillations can be avoided by using individual task performance information. Let the exponential moving average ( $\bar{f}_k^i$ ) of the action of the individual  $i$  be defined as

$$\bar{f}_k^i = \rho f_k^i + (1 - \rho) \bar{f}_{k-1}^i \quad (16)$$

where  $\rho$  is the same configurable parameter of (12).  $\bar{f}_k^i \in [0, 1]$  because  $f_k^i$  is a binary variable.  $\bar{f}_k^i$  takes the value 0 or 1 when an individual is specialized because it tends to do the same action, while  $\bar{f}_k^i \rightarrow 0.5$ , when there is high variability of the individual task performance.

Let the stress factor be defined as

$$\eta_k = 4 \cdot \bar{f}_k^i (1 - \bar{f}_k^i) \quad (17)$$

where  $\eta_k \in [0, 1]$ . For the individuals that are not performing an oscillatory behavior  $\eta_k \rightarrow 0$  and for the individuals that

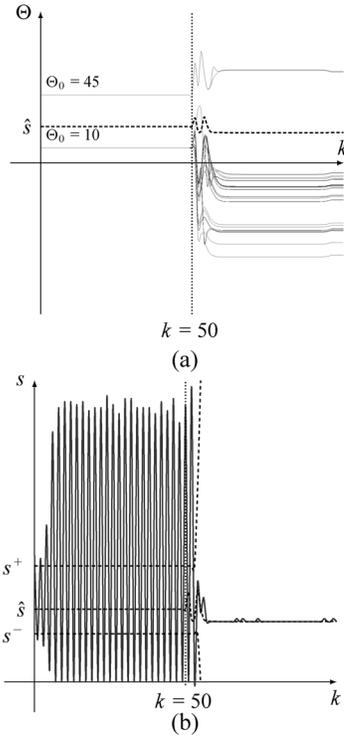


Fig. 6. Variable threshold algorithm example with stress factor for  $s_0 \notin A(\hat{s})$  with  $D = 0$ ,  $\delta = 100$ , and  $N = 100$  with 80% of individuals having  $\Theta_0 = 10$  and 20% having  $\Theta_0 = 45$ . (a) Evolution in time of response thresholds. (b) Evolution of stimulus intensity and domain of attraction limits ( $s^-$ ,  $s^+$ ).

have converged to a periodic equilibrium point  $\eta_k \rightarrow 1$ . The stress factor can be used to avoid an oscillatory behavior if the most stressed individuals invert the process of specialization. So that (10) is modified as

$$\Delta\Theta_k^i = \begin{cases} -(1 - 2 \cdot \eta_k) \cdot \varphi_k + \xi_k & \text{if } f_k^i = 1 \\ (1 - 2 \cdot \eta_k) \cdot \varphi_k - \xi_k & \text{if } f_k^i = 0 \end{cases} \quad (18)$$

where  $(1 - 2 \cdot \eta_k) \in [-1, 1]$  and it modifies the learning coefficient magnitude and direction.

An example of the stress factor operation is shown in Fig. 6. The environmental parameter, the number of individuals, and the response thresholds initial values are the same as the those of the example of Fig. 5. The initial stimulus intensity value is in the domain of attraction of the oscillatory equilibrium points. The variable threshold algorithm is activated at time step  $k = 50$ . In this case, the response thresholds spread into two groups, similar to the result of the example of Fig. 4, as is shown in Fig. 6(a). The stimulus intensity begins the example of oscillating and it converges to a nonoscillatory situation when the variable threshold algorithm is activated with the stress factor [see Fig. 6(b)].

### C. Reference Stimulus Intensity

Although the dynamical system can achieve a stable and nonoscillatory situation, the attractor value is uncertain. This value depends on the initial conditions and response threshold evolution. Once the specialization mechanism has converged, individuals continue performing the same behavior. Therefore,

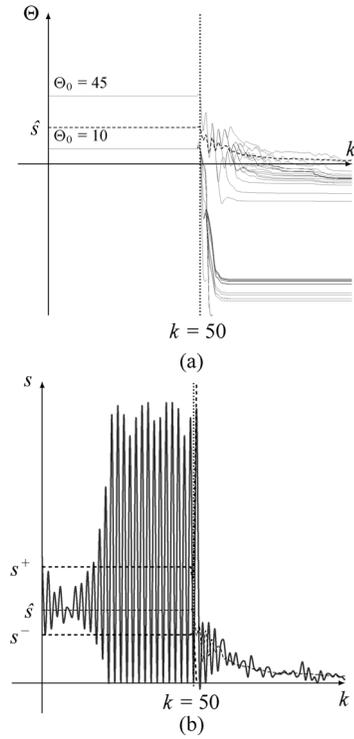


Fig. 7. Variable threshold algorithm example with stress factor and reference stimulus intensity  $R = 0$  for  $s_0 \notin A(\hat{s})$  with  $D = 0$ ,  $\delta = 100$ , and  $N = 100$  with 80% of individuals having  $\Theta_0 = 10$  and 20% having  $\Theta_0 = 45$ . (a) Evolution in time of response thresholds. (b) Evolution of stimulus intensity and domain of attraction limits ( $s^-$ ,  $s^+$ ).

to modify the attractor value, the response threshold variation has to modify the specialization process. Let  $\lambda_k$  be the response threshold variation to modify the attractor value

$$\lambda_k = \begin{cases} -(1 - \eta_k) \cdot (1 - e^{-|S_k - R|}) & \text{if } S_k > R \\ (1 - \eta_k) \cdot (1 - e^{-|S_k - R|}) & \text{if } S_k < R \end{cases} \quad (19)$$

where  $\lambda_k \in [-1, 1]$  and  $R$  is the reference stimulus intensity which is the desired attractor value. Equation 19 satisfies that if  $S_k > R \Rightarrow \lambda_k \rightarrow -1$ , if  $S_k < R \Rightarrow \lambda_k \rightarrow 1$  and if  $S_k \approx R \Rightarrow \lambda_k \rightarrow 0$ .  $\lambda_k$  displaces the response thresholds to modify the attractor value, but it can be in conflict with the stress factor and the over-specialization process. For this reason, in order to incorporate the modification of the attractor value in the variable threshold algorithm, 18 must be redefined as

$$\Delta\Theta_k^i = \begin{cases} (2(\lambda_k + \eta_k) - 1) \cdot \varphi_k + (\lambda_k + 1)\xi_k & \text{if } f_k^i = 1 \\ (2(\lambda_k - \eta_k) + 1) \cdot \varphi_k + (\lambda_k - 1)\xi_k & \text{if } f_k^i = 0 \end{cases} \quad (20)$$

An example of how  $s_k$  converges to  $R = 0$  is shown in Fig. 7. The environmental parameter, the number of individuals, and the response thresholds initial values are the same as those of the example of Fig. 6. The variable threshold algorithm is activated at time step  $k = 50$ . The response thresholds tend to take lower values, modifying the asymptotically stable equilibrium point, as shown in Fig. 7(a). The evolution of the stimulus intensity performs a nonoscillatory behavior and converges to the reference value, as can be observed in Fig. 7(b).

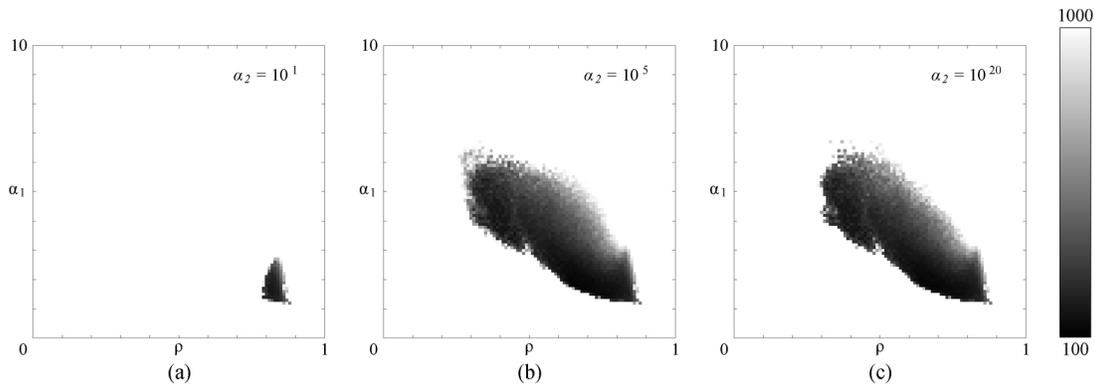


Fig. 8. Grayscale color-map where the darker the shorter convergence time. It represents the  $Q_3$  of convergence time for the exhaustive search of the configurable parameters of the variable threshold algorithm (see Section III). (a) Scanning of  $\rho$  and  $\alpha_1$  for  $\alpha_2 = 10^1$ . (b) Scanning of  $\rho$  and  $\alpha_1$  for  $\alpha_2 = 10^5$ . (c) Scanning of  $\rho$  and  $\alpha_1$  for  $\alpha_2 = 10^{20}$ .

#### IV. EXPERIMENTS

In this section, we propose a set of experiments to optimize the variable threshold parameters and to test them with different populations. As described in Section III, the variable threshold algorithm has three configurable parameters:  $\rho$ ,  $\alpha_1$ , and  $\alpha_2$ . They modify how the variable threshold algorithm adapts to the environment to achieve the previously defined dynamic behavior. This algorithm can be optimized following a certain criterion. An exhaustive search of these parameters has been carried out to locate the minimum convergence time.

The response threshold function used for the experiments is the Sigmoid function

$$T(s_k, \Theta^i) = \frac{1}{1 + e^{-\beta(s_k - \Theta^i)}} \quad (21)$$

where  $\beta = -0.2$  for all individuals. Using this response threshold function, simulations with different environment and initial configurations have been performed. In the exhaustive search, every combination of  $\rho$ ,  $\alpha_1$ , and  $\alpha_2$  has been clustered in a set. Every set of these parameters has been simulated for populations of  $10^2$ ,  $10^3$ , and  $10^4$  individuals. Moreover, each of these populations has been simulated for two values of coefficient  $D$ , 0 and 1.  $\delta$  has been fixed to  $N/2$ . The reference stimulus intensity has been fixed to  $R = 0$ . The initial stimulus intensity value has been set to  $s_0 = \delta$ . Initial response threshold values have also been tested, initializing the response thresholds with two uniform probability distributions between  $[-N, 0]$  and  $[0, N]$ . These distributions have been chosen to provoke asymmetric situations with respect to  $R$ . Moreover, because of the pseudo random nature of the experiments, each configuration has been simulated for 30 different seeds of a random generator number. Therefore, 360 simulations have been performed for every set of  $\rho$ ,  $\alpha_1$ , and  $\alpha_2$ , calculating the convergence time ( $T_C$ ) for every simulation.

It is considered that the system has converged when the following two constraints are satisfied: 1) The stimulus intensity mean or second quartile ( $Q_2$ ) is between  $[R-1, R+1]$ , the first quartile ( $Q_1$ ) is between  $[R-2, R]$ , and the third quartile ( $Q_3$ ) is between  $[R, R+2]$ . This convergence constraint operates as a mean filter, removing the noise of the stochastic process and ensuring the convergence to the reference stimulus intensity  $R$ .

2) The stimulus intensity variance is in  $[0.5, 2.0]$ . The lower limit indicates that the individuals are not overspecialized and the upper one indicates a minimum distance between the response thresholds and the attractor.

To analyze the effects of the configurable parameters on  $T_C$ , several sets of these parameters have been simulated. Every set is a different combination of  $\rho$ ,  $\alpha_1$ , and  $\alpha_2$ . Each parameter is discretized in a certain range with an interval between values, such that,  $\rho \in [0, 1]$  with an interval of 0.01,  $\alpha_1 \in [0, 10]$  with an interval of 0.1 and  $\alpha_2 \in [1, 10^{20}]$  with a logarithmic interval of 10. This combination produces  $2 \cdot 10^5$  different sets of configurable parameters. As aforementioned, 360 simulations have been performed for every set and this leads to  $72 \cdot 10^6$  simulations for the exhaustive search campaign.

The third quartile of the convergence time is shown in Fig. 8 with color maps. Each color map has been plotted for a fixed value of  $\alpha_2$  and represents the scanning of  $\rho$  and  $\alpha_1$ . The values of the parameters and the convergence time have a nonlinear relationship because the shape of the color map. Although  $\alpha_2$  has been simulated for more values than shown, the more representative ones have been selected. The optimum value of  $\alpha_2$  is  $10^5$  whose color map is shown in Fig.8(b). A lower value ( $\alpha_2 = 10^1$ ) and a upper value ( $\alpha_2 = 10^{20}$ ) than the optimum one are plotted in figures 8(a) and (c), respectively. The minimum convergence time in  $Q_3$  is found for  $\rho = 0.64$ ,  $\alpha_1 = 2.0$  and  $\alpha_2 = 10^5$ .

Finally, a study of the effects on convergence time ( $T_C$ ) for different population sizes has been carried out. For each population size, the study has been divided for  $D = 0$  and  $D \approx 1$ , which is a set  $\{N, D\}$ . For each set  $\{N, D\}$ , several simulations have been carried out. The response threshold functions,  $\delta$ ,  $s_0$ , and  $R$  in each simulation are the same as the exhaustive search. Different initial response threshold values have also been simulated with uniform distributions between  $[-N, 0]$  and  $[0, N]$ . The number of seeds of the random number generator has been set to 100, to have enough samples for a statistical comparison. Therefore, there are 400 simulations for every set  $\{N, D\}$ . The results of these simulations are shown in Fig. 9. The convergence times of all studied sets  $\{N, D\}$  are represented through box-plots. The  $T_C$  increases approximately linearly with a logarithmic

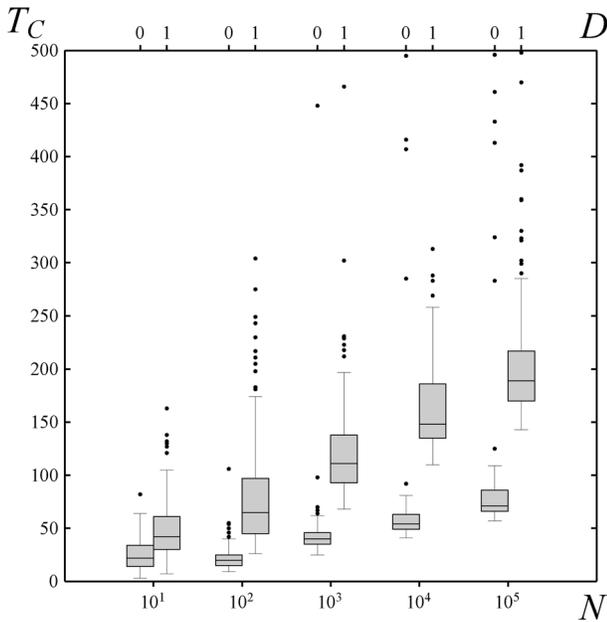


Fig. 9. Box-plots of population study. Each population is simulated for  $D = 0$  and  $D = 1$  making a set  $\{N, D\}$  represented in the  $x$ -axis. The  $y$ -axis represents the convergence time ( $T_C$ ). Each box comprises observations ranging from the first to the third quartile. The median is indicated by a horizontal bar, dividing the box into the upper and lower part. The whiskers extend to the farthest data points that are within 1.5 times the interquartile range. Outliers are shown as points.

population increase; it means that variable threshold algorithm is scalable.

## V. CONCLUSION

In this paper, we have shown how dynamical systems theory can be used for the analysis and design of division of labor models and algorithms. Depending on the complexity of the model, the dynamic properties can be defined to ensure dynamic behaviors, stability, or design rules. In our case, we have analyzed a response threshold model for the division of labor as a discrete-time nonlinear dynamical system, defining five properties. The response threshold model has been transformed into an analytical one based on the average of the individual task performances. The structure of the dynamical system does not vary in time because the response thresholds are fixed. The response threshold functions, the response threshold values and the environment configuration define two possible dynamical behaviors, asymptotically stable and oscillatory.

We have also proposed and implemented a variable threshold algorithm. The algorithm modifies the equilibrium points in real time in order to force an asymptotically stable behavior of the response threshold model. The operation of the variable threshold algorithm is based on a specialization mechanism. We have shown how the proposed specialization equations increase stability and solve the problem of over-specialization. Moreover, oscillatory behaviors are avoided by the use of a stress factor. It also enables the system to converge asymptotically to a specific stable reference value. Finally, the algorithm has been tested in different environments and with different

numbers of individuals. All experiments behaved as deduced from the analytical study.

This implementation of the variable threshold algorithm allows the use of response threshold models as distributed controllers. In this case, each individual can be considered as a single controller, with no need of explicit information exchange between them. This type of algorithms can be used for applications where the communication is costly or noisy, such as large data networks or satellite synchronization. They can also be used for applications where each element cannot obtain information from the others and only global information is available, e.g., the electricity grid or traffic control. Another potential use is for situations where information obtained by individuals may be noisy or corrupted, such as in multirobot applications.

A potential direction for future research would be an algorithm that allows individuals to work in a multiple task environment, where individuals have to choose the task they have to work on. This environment needs a multivariable dynamical system study. Another possible direction is to increase the complexity of the environment, for example modifying the  $\delta$  environment parameter for a time-dependent signal  $\delta(t)$ . In this situation, the algorithm must be able to adapt to a dynamical environment which is of great interest in the design of dynamical controllers for real environment applications.

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**Manuel Castillo-Cagigal** received the M.Sc. degree in telecommunication engineering from the Universidad de Málaga (UMA), Málaga, Spain, in 2009, and the M.Sc. degree in photovoltaic solar energy from the Universidad Politécnica de Madrid (UPM), Madrid, Spain, in 2010, where he is currently a Ph.D. student in the area of control and energy management with ETSI de Telecomunicación.

In 2013, he was a Visiting Researcher with the Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle, Université Libre de Bruxelles, Brussels, Belgium.

He has authored and co-authored five international journal papers and eight conference proceeding. His current research interests include swarm intelligence, control systems, sensors networks, and demand-side management applications specially focused on smart-grids and renewable energies.



**Eduardo Matallanas** received the M.Sc. degree in electronic engineering and the M.Sc. degree in photovoltaic solar energy from the Universidad Politécnica de Madrid, Madrid, Spain, in 2010 and 2011, respectively, where he is currently pursuing the Ph.D. degree in the area of control and energy management with ETSI de Telecomunicación and Instituto de Energía Solar.

He has authored and co-authored four international journal papers and five refereed conference proceeding.

His current research interests include swarm intelligence, artificial intelligence, neuroscience, control systems, sensors networks, and demand-side management applications especially focused on smart grids and renewable energies.



**Iñaki Navarro** was born in Madrid, Spain, in 1980. He received the M.Sc. degree in telecommunication engineering and the Ph.D. degree in robotics and automation from Universidad Politécnica de Madrid (UPM), Madrid, in 2004 and 2010, respectively.

In 2012, he was a Post-Doctoral Researcher with ETSI de Telecomunicación, UPM. Since 2013, he has been a Post-Doctoral Fellow with the Distributed Intelligent Systems and Algorithms Laboratory, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland. He has authored and co-authored various international journal articles, and several book chapters and refereed conference papers. His current research interests include formation control of mobile robots, swarm robotics, distributed consensus of agents and mobile robotics, sensor monitoring, machine learning, and robot learning.



**Estefanía Caamaño Martín** received the degree in telecommunications engineering and the Ph.D. degree in telecommunications engineering from the Universidad Politécnica de Madrid (UPM), Madrid, Spain, in 1994 and 1998, respectively.

She is a Researcher with Solar Energy Institute, UPM, in the field of photovoltaic systems, and is an Associate Professor of electronics technology with ETSI de Telecomunicación, UPM. She is an author and co-author of 12 book chapters, 15 refereed journal papers, and over 45 conference proceeding

papers. Her current research interests include photovoltaic distributed generation especially focused on urban environments from architectural and electrical energy perspectives.



**Félix Monasterio-Huelin** received the Ph.D. degree from the Universidad Politécnica de Madrid (UPM), Madrid, Spain, in 1987.

He is currently a Professor with the ETSI de Telecomunicación, UPM. His current research interests include field of intelligent robotics, neural networks, automatic control, and haptic devices.



**Álvaro Gutiérrez** (S'99–M'08–SM'12) received the M.Sc. degree in electronic engineering and the Ph.D. degree in computer science from the Universidad Politécnica de Madrid (UPM), Madrid, Spain, in 2004 and 2009, respectively.

He was a Visiting Researcher with the Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle, Université Libre de Bruxelles, Brussels, Belgium in 2008. He is currently an Associate Professor of automatic and control systems with ETSI de Telecomunicación, UPM. He has authored and co-authored three book chapters and over 35 refereed journal papers and conference proceeding papers. His current research interests include swarm robotics, control systems, sensor networks, and demand-side management applications specially focused on smart-grids and renewable energies.