From _Control System Design_
by Karl Johan Åström, 2002

$$G(s) = \frac{a}{s+a}$$

Table 7.1 Properties of the response to reference values for the first order system $G_{xr} = a/(s+a)$.

Propety	*Value*
Rise time	$T_r = 1/a = T$
Delay time	$T_d = 0.69/a = 0.69T$
Settling time (2%)	$T_s = 4/a = 4T$
Overshoot	$o = 0$
Error coefficients	$e_0 = 0,\ e_1 = 1/a = T$
Bandwidth	$\omega_b = a$
Resonance peak	$\omega_r = 0$
Sensitivities	$M_s = M_t = 1$
Gain margin	$g_m = \infty$
Phase margin	$\varphi_m = 90°$
Crossover frequency	$\omega_{gc} = a$
Sensitivity frequency	$\omega_{sc} = \infty$

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

The system has two poles, they are complex if $\zeta < 1$ and real if $\zeta > 1$. The step response of the system is

$$h(t) = \begin{cases} 1 - \dfrac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) & \text{for } |\zeta| < 1 \\ 1 - (1 + \omega_0 t)e^{-\omega_0 t} & \text{for} \zeta = 1 \\ 1 - \Big(\cosh \omega_d t + \dfrac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \omega_d t\Big) e^{-\zeta\omega_d t} & \text{for } |\zeta| > 1 \end{cases}$$

where $\omega_d = \omega_0\sqrt{|1-\zeta^2|}$ and $\phi = \arccos\zeta$. When $\zeta < 1$ the step response is a damped oscillation, with frequency $\omega_d = \omega_0\sqrt{1-\zeta^2}$. Notice that the step response is enclosed by the envelopes

$$e^{-\zeta\omega_0 t} \le h(t) \le 1 - e^{-\zeta\omega_0 t}$$

This means that the system settles like a first order system with time constant $T = \frac{1}{\zeta\omega_0}$. The 2% settling time is thus $T_s \approx \frac{4}{\zeta\omega_0}$. Step responses for different values of ζ are shown in Figure 4.9.

The maximum of the step response occurs approximately at $T_p \approx \pi/\omega_d$, i.e. half a period of the oscillation. The overshoot depends on the damping. The largest overshoot is 100% for $\zeta = 0$. Some properties of the step response are summarized in Table 7.3.

The system (7.10) can be interpreted as a feedback system with the loop transfer function

$$L(s) = \frac{\omega_0^2}{s(s + 2\zeta\omega_0)}$$

This means that we can compute quantities such as sensitivity functions and stability margins. These quantities are summarized in Table 7.3.

Table 7.3 Properties of the response to reference values of a second order system.

Property	*Value*
Rise time	$T_r = \omega_0 e^{\phi/\tan\phi} \approx 2.2 T_d$
Delay time	T_d
Peak time	$T_p \approx \pi/\omega_D = T_d/2$
Settling time (2%)	$T_s \approx 4/(\zeta\omega_0)$
Overshoot	$o = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
Error coefficients	$e_0 = 0,\ e_1 = 2\zeta/\omega_0$
Bandwidth	$\omega_b = \omega_0\sqrt{1-2\zeta^2+\sqrt{(1-2\zeta^2)^2+1}}$
Maximum sensitivity	$M_s = \sqrt{\frac{8\zeta^2+1+(4\zeta^2+1)\sqrt{8\zeta^2+1}}{8\zeta^2+1+(4\zeta^2-1)\sqrt{8\zeta^2+1}}}$
Frequency	$w_{ms} = \frac{1+\sqrt{8\zeta^2+1}}{2}\omega_0$
Max. comp. sensitivity	$M_t = \begin{cases} 1/(2\zeta\sqrt{1-\zeta^2}) & \text{if } \zeta \le \sqrt{2}/2 \\ 1 & \text{if } \zeta \le \sqrt{2}/2 \end{cases}$
Frequency	$\omega_{mt} = \begin{cases} \omega_0\sqrt{1-2\zeta^2} & \text{if } \zeta \le \sqrt{2}/2 \\ 1 & \text{if } \zeta \le \sqrt{2}/2 \end{cases}$
Gain margin	$g_m = \infty$
Phase margin	$\varphi_m = 90^\circ - \arctan\omega_c/(2\zeta\omega_0)$
Crossover frequency	$\omega_{gc} = \omega_0\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$
Sensitivity frequency	$\omega_{sc} = \omega_0/\sqrt{2}$

7.8 Relations Between Specifications

A good intuition about the different specifications can be obtained by investigating the relations between specifications for simple systems as is given in Tables 7.1, 7.2 and 7.3.

The Rise Time Bandwidth Product

Consider a transfer function $G(s)$ for a stable system with $G(0) \neq 0$. We will derive a relation between the rise time and the bandwidth of a system. We define the rise time by the largest slope of the step response.

$$T_r = \frac{G(0)}{\max_t \boldsymbol{g}(t)} \tag{7.12}$$

where $\boldsymbol{g}$ is the impulse response of G, and let the bandwidth be defined as

$$\omega_b = \frac{\int_0^\infty |G(i\omega)|}{\pi G(0)} \tag{7.13}$$

This implies that the bandwidth for the system $G(s) = 1/(s+1)$ is equal to 1, i.e. the frequency where the gain has dropped by a factor of $1/\sqrt{2}$. The impulse response $\boldsymbol{g}$ is related to the transfer function G by

$$\boldsymbol{g}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}G(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}G(i\omega)d\omega$$

Hence

$$\max_t \boldsymbol{g}(t) \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{i\omega t}G(i\omega)\right|d\omega = \frac{1}{\pi}\int_0^\infty |G(i\omega)|d\omega$$

Equations (7.12) and (7.13) now give

$$T_r\omega_b \geq 1$$

This simple calculation indicates that the product of rise time and bandwidth is approximately constant. For most systems the product is around 2.

Bode's Relations

$$
\begin{aligned}
\arg G(i\omega_0) &= \frac{2\omega_0}{\pi}\int_0^\infty \frac{\log|G(i\omega)| - \log|G(i\omega_0)|}{\omega^2-\omega_0^2}d\omega \\
&= \frac{1}{\pi}\int_0^\infty \frac{d\log|G(i\omega)|}{d\log\omega}\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d
\end{aligned}
$$

$$
\approx \frac{\pi}{2}\frac{d\log|G(i\omega)|}{d\log\omega}
$$

$$
\begin{aligned}
\frac{\log|G(i\omega)|}{\log|G(i\omega_0)|} &= -\frac{2\omega_0^2}{\pi}\int_0^\infty \frac{\omega^{-1}\arg G(i\omega) - \omega_0^{-1}\arg G(i\omega_0)}{\omega^2-\omega_0^2}d\omega \\
&= -\frac{2\omega_0^2}{\pi}\int_0^\infty \frac{d(\omega^{-1}\arg G(i\omega))}{d\omega}\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d\omega
\end{aligned}
\tag{3.31}
$$

The formula for the phase tells that the phase is a weighted average of the logarithmic derivative of the gain, approximatively

$$
\arg G(i\omega) \approx \frac{\pi}{2}\frac{d\log|G(i\omega)|}{d\log\omega} \tag{3.32}
$$

This formula implies that a slope of +1 corresponds to a phase of $\pi/2$, which holds exactly for the differentiator, see Figure 3.17. The exact formula (3.31) says that the differentiated slope should be weighted by the kernel

$$
\int_0^\infty \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d\omega = \frac{\pi^2}{2}
$$

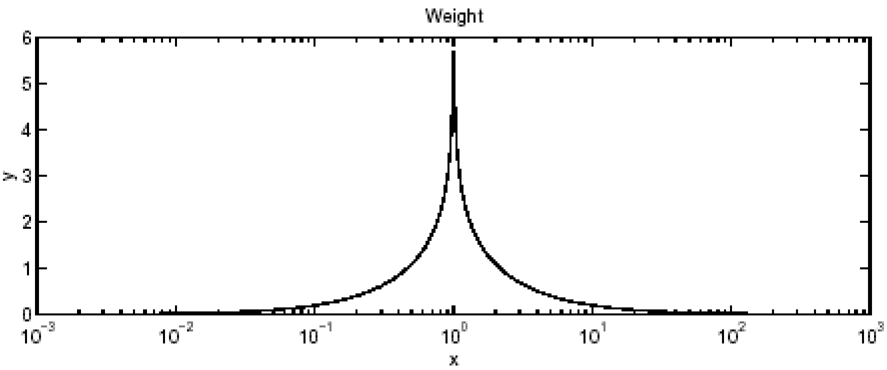

Figure 3.23 The weighting kernel in Bodes formula for computing the phase from the gain.